On adelic model of boson Fock space

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We construct a canonical embedding of the Schwartz space on $\mathbb{R}^n$ to the space of distributions on the adelic product of all the p-adic numbers. This map is equivariant with respect to the action of the symplectic group $\mathrm{Sp}(2n,\mathbb{Q})$ over rational numbers and with respect to the action of rational Heisenberg group.

These notes contain two elements. First, we give a fanny funny???? realization of a space of complex functions of a real variable as a space of functions of p-adic variable. Secondly, we try to clarify classical contstruction of modular forms through θ-functions and Howe duality.

1. Introduction

1.1. Fields and rings. Below $\mathbb{Q}$ denotes the rational numbers, $\mathbb{Z}$ is the ring of integers, $\mathbb{Q}_p$ is the field of p-adic numbers, $\mathbb{Z}_p \subset \mathbb{Q}_p$ is the ring of p-adic integers. We denote the norm on $\mathbb{Q}_p$ by $|\cdot|$.

1.2. Adeles, (see [2], [11], [8]). An *adele* is a sequence

$$(a_\infty, a_2, a_3, a_5, a_7, a_{11}, \dots), \tag{1.1}$$

where $a_\infty \in \mathbb{R}$, $a_p \in \mathbb{Q}_p$ (p is a prime) and $|a_p| = 1$ for all p except a finite a number of primes.

Our main object is the ring

$$\mathbb{A} \subset \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \dots$$

consisting of the sequences

$$a = (a_2, a_3, a_5, a_7, a_{11}, \dots)$$

satisfying the same conditions. The addition and multiplication in $\mathbb{A}$ are defined coordinate-wise. Below the term *"adeles"* means the ring $\mathbb{A}$. The space of sequences of the form (1.1) we denote by $\mathbb{R} \times \mathbb{A}$.

1.3. Convergence in $\mathbb{A}$. A sequence $a^{(j)}$ in $\mathbb{A}$ converges to $a \in A$ iff

a) There is a finite set S of primes such that $|a_p^{(j)}| = 1$ for all $p \notin S$ for all j.

b) For each p, the sequence $a_p^{(j)}$ converges in $\mathbb{Q}_p$.

The image of the diagonal embedding $\mathbb{Q} \to \mathbb{A}$

$$r \mapsto (r, r, r, \dots)$$

is dense in $\mathbb{A}$.

1.4. Integration. We define the Lesbegue measure da on the ring $\mathbb{A}$ by two assumptions:

– the measure on $\prod_p \mathbb{Z}_p$ is the product-measure

– the measure is translation-invariant.

This measure is σ-finite. We define the space $L^2(\mathbb{A}^n)$ in the usual way. The Bruhat space $\mathcal{B}(\mathbb{A}^n)$ defined below is dense in $L^2(\mathbb{A}^n)$.

1.5. Adelic exponents. For an adele $a \in \mathbb{A}$, we define its exponent $\exp(2\pi i a) \in \mathbb{C}$ by

$$\exp(2\pi i a) = \prod_p \exp(2\pi i a_p).$$

[1]supported by grant NWO–047.017.015

all the factors are roots of unity, only finite number of factors is $\neq 1$.

1.6. Lattices. A *lattice* L in a $\mathbb{Q}$-linear space $\mathbb{Q}^n$ is an arbitrary additive subgroup isomorphic $\mathbb{Z}^n$. Equivalently, a lattice is a group $L \subset \mathbb{Q}^n$ having a form $\bigoplus \mathbb{Z} f_j$, where f_j is a basis in $\mathbb{Q}^n$. A *dual lattice* $L^\Diamond$ consists of $y \in \mathbb{Q}^n$, such that $\sum x_j y_j \in \mathbb{Z}$ for all the $x \in L$.

A *lattice* in the p-adic linear space $\mathbb{Q}_p^n$ is a set of the form $\bigoplus \mathbb{Z}_p f_j$, where f_j is a basis in $\mathbb{Q}_p^n$. The *standard lattice* is the set $\mathbb{Z}_p^n$.

A *lattice* in the adelic module $\mathbb{A}^n$ is a set of a form $\bigoplus_p L_p$, where $L_p \subset \mathbb{Q}_p^n$ are lattices, and L_p are the standard lattices for all p except a finite set.

For a lattice $L \subset \mathbb{Q}^n$, consider its closure $\overline{\overline{L}} \subset \mathbb{A}^n$. It is a lattice, and moreover the map $L \mapsto \overline{\overline{L}}$ is a bijection of the set of all the lattices in $\mathbb{Q}^n$ and the set of all the lattices in $\mathbb{A}^n$.

1.7. Bruhat test functions and distributions on $\mathbb{A}^n$. A *test function* f on $\mathbb{Q}_p^n$ or on $\mathbb{A}^n$ is a compactly supported locally constant complex-valued function. The Bruhat space $\mathcal{B}(\mathbb{Q}_p^n)$ (resp. $\mathcal{B}(\mathbb{A}^n)$) is the space of all the test functions.

A *distribution* is a linear functional on $\mathcal{B}(\mathbb{Q}_p^n)$ (resp. $\mathcal{B}(\mathbb{A}^n)$). We denote the space of all the distributions by $\mathcal{B}'(\mathbb{Q}_p^n)$ (resp. $\mathcal{B}'(\mathbb{A}^n)$).

1.8. The second description of the spaces $\mathcal{B}$. Let S be a subset in $\mathbb{Q}_p^n$ or $\mathbb{A}^n$. Denote by $\mathcal{I}_S$ the indicator function of S, i.e.

$$\mathcal{I}_S(x) = \begin{cases} 1, \text{ if } x \in S \\ 0, \text{ if } x \in S. \end{cases}$$

For a lattice L and a vector a, the function $\mathcal{I}_{L+a}$ is a test function. Each test function is a linear combination of functions of this type.

1.9. Third description of the spaces $\mathcal{B}$. Consider the space $\mathbb{Q}_p^n$ or $\mathbb{A}^n$. Let $K \subset L$ be lattices. Denote by $\mathcal{B}(L|K)$ the space

a) $f = 0$ outside L.

b) f is K-invariant.

The dimension of this space is the order of the quotient group L/K, in particular the dimension is finite.

Then

$$\mathcal{B}(\mathbb{Q}_p^n) = \bigcup_{K \subset L \subset \mathbb{Q}_p^n} \mathcal{B}(L|K; \mathbb{Q}_p), \qquad \mathcal{B}(\mathbb{A}^n) = \bigcup_{K \subset L \subset \mathbb{A}^n} \mathcal{B}(L|K; \mathbb{A}).$$

1.10. The space $\mathcal{M}(\mathbb{Q}^n)$. We repeat literally the previous definition. For two lattices $K \subset L \subset \mathbb{Q}^n$, denote by $\mathcal{M}(L|K)$ the space of K-invariant functions on $\mathbb{Q}^n$ supported by L. We assume

$$\mathcal{M}(\mathbb{Q}^n) = \bigcup_{K \subset L \subset \mathbb{Q}^n} \mathcal{M}(L|K).$$

The space $\mathcal{M}(\mathbb{Q}^n)$ is generated by the indicator functions $\mathcal{I}_{L+a}$ of shifted lattices.

1.11. The bijection $\mathcal{M}(\mathbb{Q}^n) \leftrightarrow \mathcal{B}(\mathbb{A}^n)$.

Proposition 1.1. *a) Each function $f \in \mathcal{M}(\mathbb{Q}^n)$ admits a unique continuous extension $\overline{\overline{f}}$ to a function on $\mathbb{A}^n$*

b) The map $f \mapsto \overline{\overline{f}}$ is a bijection $\mathcal{M}(\mathbb{Q}^n) \to \mathcal{B}(\mathbb{A}^n)$.

The statement is trivial. More constructive variant of this is statements is

$$\overline{\overline{\mathcal{I}_{L+a}}} = \mathcal{I}_{\overline{\overline{L}}+a}.$$

1.12. Space $\mathcal{P}(\mathbb{R}^n)$ of Poisson distributions. Denote by $\mathcal{S}(\mathbb{R}^n)$ the *Schwartz space* on $\mathbb{R}^n$, i.e. the space of smooth functions f on $\mathbb{R}^n$ satisfying the condition: for each $\alpha_1, \dots, \alpha_n$, and each N

$$\lim_{x\to\infty} \Big(\sum x_j^2\Big)^N \frac{\partial^{\alpha_1}}{\partial^{\alpha_1} x_1} \dots \frac{\partial^{\alpha_n}}{\partial^{\alpha_n} x_n} f(x) = 0.$$

By $\mathcal{S}'(\mathbb{R}^n)$ denote the space dual to $\mathcal{S}(\mathbb{R}^n)$, i.e., the space of all *tempered distributions* on $\mathbb{R}^n$.

Now we intend to define a certain subspace $\mathcal{P}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. This space is spanned by functions

$$\sum_{k_1,\dots,k_n\in\mathbb{Z}} \delta\big(x - \sqrt{2\pi}(b + \sum_j k_j a_j)\big),$$

where $a_1, \dots, a_n \in \mathbb{Q}^n$ are linear independent, $b \in \mathbb{Q}^n$.

Lemma 1.2. *A countable sum ψ of δ-functions is an element of $\mathcal{P}(\mathbb{R}^n)$ iff there are two lattices $K \subset L \subset \mathbb{Q}^n$ such that ψ is supported by $\sqrt{2\pi}L$ and ψ is $\sqrt{2\pi}K$-invariant.*

We denote by $\mathcal{P}(L|K) \subset \mathcal{P}(\mathbb{R}^n)$ the space of all the distributions satisfying this Lemma.

1.13. Canonical bijection $\mathcal{M}(\mathbb{Q}^n) \leftrightarrow \mathcal{P}(\mathbb{R}^n)$. Define a canonical bijective map $I_\mathbb{R} : \mathcal{M}(\mathbb{Q}^n) \to \mathcal{P}(\mathbb{R}^n)$. Let $f \in \mathcal{M}(\mathbb{Q}^n)$, let $M \subset L \subset \mathbb{Q}^n$ be corresponding lattices. We define the distribution $I_\mathbb{R} f \in \mathcal{P}(\mathbb{R}^n)$ as

$$I_\mathbb{R} f(x) = \sum_{\xi\in L} f(\xi)\delta(x - \sqrt{2\pi}\xi).$$

We obtain the bijection $\mathcal{M}(\mathbb{Q}^n) \to \mathcal{P}(\mathbb{R}^n)$. Also, for each rational lattices $K \subset L$, we have a bijection

$$\mathcal{M}(L|K) \longleftrightarrow \mathcal{P}(L|K).$$

1.14. Observation. Thus we have the canonical bijection

main
$$J_{\mathbb{R}\mathbb{A}} : \Big\{\text{space } \mathcal{P}(\mathbb{R}^n)\Big\} \longleftrightarrow \Big\{\text{adelic space } \mathcal{B}(\mathbb{A}^n)\Big\}. \tag{1.2}$$

In particular, we have canonical embeddings

$$\mathcal{S}(\mathbb{R}^n) \to \mathcal{B}'(\mathbb{A}^n),$$
$$\mathcal{B}(\mathbb{A}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

1.15. The image of the Schwartz space in $\mathcal{B}'(\mathbb{A}^n)$.

Proposition 1.3. *For $f \in \mathcal{S}(\mathbb{R}^n)$, the corresponding element $F \in \mathcal{B}'(\mathbb{A}^n)$ is*

adelic-delta
$$F(a) = \sum_{\xi\in\mathbb{Q}^n} f(\xi)\delta_\mathbb{A}(a - \xi) \tag{1.3}$$

where $\delta_\mathbb{A}$ is the adelic delta-function.

Proof. Let $L \subset \mathbb{Q}$ be a lattice, $b \in \mathbb{Q}^n$. The value of the adelic distribution $F \in \mathcal{B}'(\mathbb{A}^n)$ on the adelic test function $\mathcal{I}(\overline{\overline{L}} + b)$ is

$$\sum_{\xi\in\left[\mathbb{Q}^n\cap(\overline{\overline{L}}+b)\right]} f(\xi) = \sum_{\xi\in(L+b)} f(\xi)$$

The last expression is the value of the Poisson distribution $I_{\mathbb{R}}\mathcal{I}_{L+a}$ on the Schwartz function f. □

1.16. Result of the paper. The space $\mathcal{S}(\mathbb{R}^n)$ is equipped with the canonical action of the real Heisenberg group[2] $\mathrm{Heis}_n(\mathbb{R})$ and the real symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ (in this sense, $\mathcal{S}(\mathbb{R}^n)$ is a *bosonic Fock space* mentioned in the title).

The space $\mathcal{B}(\mathbb{A}^n)$ is equipped with the canonical action of the adelic Heisenberg group $\mathrm{Heis}_n(\mathbb{A})$ and the adelic symplectic group $\mathrm{Sp}(2n,\mathbb{A})$.

There are canonical embeddings

$$\mathrm{Heis}_n(\mathbb{Q}) \to \mathrm{Heis}_n(\mathbb{R}), \qquad \mathrm{Heis}_n(\mathbb{Q}) \to \mathrm{Heis}_n(\mathbb{A}),$$
$$\mathrm{Sp}(2n,\mathbb{Q}) \to \mathrm{Sp}(2n,\mathbb{R}), \qquad \mathrm{Sp}(2n,\mathbb{Q}) \to \mathrm{Sp}(2n,\mathbb{A});$$

in all the cases the images are dense.

th:main **Theorem 1.4.** *a) The map $J_{\mathbb{R}\mathbb{A}}$ commutes with the action of* $\mathrm{Heis}_n(\mathbb{Q})$.

b) The map $J_{\mathbb{R}\mathbb{A}}$ commutes with the action of $\mathrm{Sp}(2n,\mathbb{Q})$.

Corollary 1.5. *For $f \in \mathcal{S}(\mathbb{R})$ denote by $\widehat{f}$ its Fourier transform. Then the adelic Fourier transform of the distribution (adelic-delta 1.3) is*

$$\mathrm{const} \cdot \sum_{\xi \in \mathbb{Q}^n} \widehat{f}(\xi)\delta_{\mathbb{A}}(a-\xi)$$

th:modular **Theorem 1.6.** *For each $f \in \mathcal{P}(\mathbb{R}^n)$, there is a congruence subgroup in* $\mathrm{Sp}(2n,\mathbb{Z})$ *that fixes f.*

1.17. Another description of the operator $J_{\mathbb{R}\mathbb{A}}$. Consider the space $\mathbb{R}^n \times \mathbb{A}^n$ (in fact, it is the adelic space in the usual sense). Consider the tensor product $\mathcal{S}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{A}^n)$, and consider the linear functional (Poisson–Weil distribution) on this space given by

$$K(x,\xi) = \sum_{\xi \in \mathbb{Q}^n} \delta_{\mathbb{R}^n}(x+\xi)\delta_{\mathbb{A}^n}(a-\xi)$$

Our operator $\mathcal{S}(\mathbb{R}^n) \to \mathcal{B}'(\mathbb{A}^n)$ is the pairing

$$f(x) \mapsto F(a) = \bigl\{K(x,a), f(x)\bigr\}$$

2. Rational Heisenberg group

2.1. Heisenberg group. By Heis_n we denote the group of $(1+n+1)\times(1+n+1)$-matrices

heis-matrix
$$R(v_+,v_-,\alpha) = \begin{pmatrix} 1 & v_+ & \alpha + \frac{1}{2}v_+v_-^t \\ 0 & 1 & v_-^t \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

Here v_+, v_- are matrices-rows, v_-^t is a matrix-column, the sign ^{t} is the transposition. We have

heis-product
$$R(v_+,v_-,\alpha)R(w_+,w_-,\beta) = R(v_++w_+, v_-+w_-, \alpha+\beta+\frac{1}{2}(v_+w_-^t - w_+v_-^t)) \tag{2.2}$$

We consider 4 Heisenberg groups, $\mathrm{Heis}_n(\mathbb{Q})$, $\mathrm{Heis}_n(\mathbb{R})$, $\mathrm{Heis}(\mathbb{Q}_p)$, $\mathrm{Heis}_n(\mathbb{A})$, this means that matrix elements of (heis-matrix 2.1) are elements of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{A}$.

[2] All the definitions are given below.

The group $\mathrm{Heis}_n(\mathbb{Q})$ is a dense subgroup in $\mathrm{Heis}_n(\mathbb{R})$, $\mathrm{Heis}_n(\mathbb{Q}_p)$ $\mathrm{Heis}_n(\mathbb{A})$.

2.2. The standard representations of Heisenberg groups. These representations are given by almost the same formulae for the rings $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{A}$, but these formulae differs by position of factors 2π.

Real case. The group $\mathrm{Heis}_n(\mathbb{R})$ acts in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ on $\mathbb{R}^n$ by the transformations

$$T_{\mathbb{R}}(v_+, v_-, \alpha) f(x) = f(\sqrt{2\pi}(x+v_+)) \exp\bigl\{\sqrt{2\pi} i x v_-^t + 2\pi i(\alpha + \frac{1}{2} w_+ v_-^t)\bigr\}.$$

This formula also defines unitary operators in $L^2(\mathbb{R}^n)$ and continuous transformations of the space $\mathcal{S}'(\mathbb{R}^n)$ of the space of tempered distributions on $\mathbb{R}^n$.

Adelic case. The group $\mathrm{Heis}_n(\mathbb{A})$ acts on the space $\mathcal{B}(\mathbb{A}^n)$ by the formula

$$T(v_+, v_-, \alpha) f(x) = f(x+v_+) \exp\bigl\{2\pi i(x v_-^t + \alpha + \frac{1}{2} w_+ v_-^t)\bigr\}. \tag{2.3}$$

This formula also defines unitary operators in $L^2(\mathbb{A}^n)$ and continuous operators in the space $\mathcal{B}'(\mathbb{A}_n)$ of adelic distributions.

p-adic case. The action of $\mathrm{Heis}_n(\mathbb{Q}_p)$ on $\mathcal{B}(\mathbb{Q}_p)$ and $\mathcal{B}'(\mathbb{Q}_p)$ is defined by the same formula.

Rational case. The group $\mathrm{Heis}_n(\mathbb{Q})$ acts in the space $\mathcal{M}(\mathbb{Q}^n)$ via the same formula (2.3).

2.3. Relations between the standard representations of $\mathrm{Heis}_n(\cdot)$.

Proposition 2.1. *a) The subgroup $\mathrm{Heis}_n(\mathbb{Q}) \subset \mathrm{Heis}_n(\mathbb{R})$ preserves the space $\mathcal{P}(\mathbb{R}^n)$.*

b) The canonical map $I_{\mathbb{R}} : \mathcal{M}(\mathbb{Q}^n) \to \mathcal{P}(\mathbb{R}^n)$ commutes with the action of $\mathrm{Heis}_n(\mathbb{Q})$.

c) The canonical bijection $\mathcal{M}(\mathbb{Q}^n) \to \mathcal{B}(\mathbb{A}^n)$ commutes with the action of $\mathrm{Heis}_n(\mathbb{Q})$.

This statement is more-or-less obvious. It also implies Theorem 1.4.a.

2.4. Irreducibility.

Lemma 2.2. *The representation of $\mathrm{Heis}_n(\mathbb{Q})$ in $\mathcal{M}(\mathbb{Q}^n)$ is irreducible. Any operator $A : \mathcal{M}(\mathbb{Q}^n) \to \mathcal{M}(\mathbb{Q}^n)$ commuting with the action of $\mathrm{Heis}_n(\mathbb{Q})$ is a multiplication by a constant.*

Proof. First, we present an alternative description of the space $\mathcal{M}(L|K)$, it consists of functions fixed with respect to operators

$$T_v f(x) = f(x+v), \qquad v \in K, \tag{2.4}$$

$$S_w f(x) = f(x) \exp(2\pi i x w^t), \qquad w \in L^\diamond. \tag{2.5}$$

The space $\mathcal{M}(L|K)$ is point-wise fixed by the group $G(L|M)$ generated by these operators.

The space $\mathcal{M}(L|K)$ is invariant with respect to the group $D(L|K)$ generated by the operators T_v, where $v \in L$, and S_w, where $w \in K^\diamond$.

Hence the quotient-group $A(L|M) = D(L|M)/G(L|M)$ acts in $\mathcal{M}(L|M)$. In fact, this group is generated by the same operators T_v, S_w, see (2.4)–(2.5), but now we consider v as an element of L/M and w as an element of $M^\diamond/L^\diamond$ (in fact, $A(L|M)$ is a finite Heisenberg group).

Let us show that the representation of $A(L|M)$ in the space $\mathcal{M}(L|M)$ is irreducible. The subgroup of $A(L|M)$ generated by the operators S_w has a simple specter, its eigenvectors are δ-functions on L/M. Hence any invariant subspace is

spanned by some collection of δ-functions. But T_v-invariance implies the triviality of an invariant subspace.

Now the both statements of the Lemma become obvious. $\square$

3. Weil representation

On the Weil representation, see [11], [7], [3], [6].

3.1. Symplectic groups. Consider a ring $\mathbb{K} = \mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Q}$, $\mathbb{A}$, $\mathbb{Z}$, $\mathbb{Z}_p$. Consider the space $\mathbb{K}^n \oplus \mathbb{K}^n$ equipped with a skew-symmetric bilinear form with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By $\mathrm{Sp}(2n, \mathbb{K})$ we denote the group of all the operators in $\mathbb{K}^n \oplus \mathbb{K}^n$ preserving this form, we write its elements as block matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

An element of the adelic symplectic group $\mathrm{Sp}(2n, \mathbb{A})$ also can be considered as a sequence $(g_2, g_3, g_5, \dots)$, where $g_p \in \mathrm{Sp}(2n, \mathbb{Q}_p)$, and $g_p \in \mathrm{Sp}(2n, \mathbb{Z}_p)$ for all the p except finite number.

3.2. Automorphisms of the Heisenberg groups. Let $\mathbb{K} = \mathbb{R}, \mathbb{Q}, \mathbb{Q}_p, \mathbb{A}$. The symplectic group $\mathrm{Sp}(2n, \mathbb{K})$ acts on the Heisenberg group $\mathrm{Heis}_n(\mathbb{K})$ by automorphisms

$$\sigma(g): \ \bigl\{v_+ \oplus v_-\bigr\} \oplus \alpha \mapsto \Bigl\{(v_+ \oplus v_-) \begin{pmatrix} A & B \\ C & D \end{pmatrix}\Bigr\} \oplus \alpha,$$

see (2.2).

3.3. Real case.

Theorem 3.1. *a) For each $g \in \mathrm{Sp}(2n, \mathbb{R})$, there is a unique up to a factor unitary operator $\mathrm{We}(g) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that for each $h \in \mathrm{Heis}_n$*

weil-def

$$T(\sigma(h)) = \mathrm{We}(g)^{-1} T(h) \mathrm{We}(g). \tag{3.1}$$

b) For each g_1, $g_2 \in \mathrm{Sp}(2n, \mathbb{R})$,

$$\mathrm{We}(g_1)\mathrm{We}(g_2) = c(g_1, g_2)\mathrm{We}(g_1 g_2),$$

where $c(g_1, g_2) \in \mathbb{C}$. Moreover, there is a choice of $\mathrm{We}(g)$, such that $c(g_1, g_2) = \pm 1$ for all g_1, g_2.

Thus $\mathrm{We}(\cdot)$ is a projective representation of $\mathrm{Sp}(2n, \mathbb{R})$. It is named the *Weil representation.*

It is easy to write the operators $\mathrm{We}(g)$ for some special matrices g,

weil-r-1

$$\mathrm{We} \begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix} f(x) = |\det(A)|^{-1/2} f(x A^{t-1}), \tag{3.2}$$

weil-r-2

$$\mathrm{We} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y) \exp\{ixy^t\}\, dy, \tag{3.3}$$

weil-r-3

$$\mathrm{We} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} f(x) = \exp\{\frac{i}{2} x B x^t\} f(x), \tag{3.4}$$

where the matric B is symmetric, $B = B^t$.

Since these elements generate the whole group $\mathrm{Sp}(2n, \mathbb{R})$, our formulae allow to obtain $\mathrm{We}(g)$ for an arbitrary $g \in \mathrm{Sp}(2n, \mathbb{R})$.

th:weil-r

Theorem 3.2. *The space $\mathcal{P}(\mathbb{R}^{2n})$ is invariant with respect to the action of the group $\mathrm{Sp}(2n, \mathbb{Q})$.*

Proof. Obviously, $\mathcal{P}(\mathbb{R}^n)$ is invariant with respect to operators (3.2), (3.4) with rational matrices A, B.

By the Poisson summation formula, $\mathcal{P}(\mathbb{R}^n)$ is invariant with respect to the Fourier transform (3.3).

It can be readily checked that the group $\mathrm{Sp}(2n,\mathbb{R})$ is generated by elements of these 3 types, and this finishes the proof. $\square$.

3.4. p-adic Weil representation. For the group $\mathrm{Sp}(2n,\mathbb{Q}_p)$, the literal analog of Theorem 3.2 is valid. In this case the operators $\mathrm{We}(g)$ are unitary in $L^2(\mathbb{Q}_p^n)$ and preserve the Bruhat space $\mathcal{B}(\mathbb{Q}_p^n)$.

Analogs of formulae (3.2)–(3.3) also can be easily written,

weil-q-1
$$\mathrm{We}\begin{pmatrix} A & 0\\ 0 & A^{t-1}\end{pmatrix} f(x) = |\det(A)|^{-1/2} f(xA^{t-1}), \tag{3.5}$$

weil-q-2
$$\mathrm{We}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} f(x) = \int_{\mathbb{R}^n} f(y)\exp\{2\pi i x y^t\}\, dy, \tag{3.6}$$

weil-q-3
$$\mathrm{We}\begin{pmatrix} 1 & B\\ 0 & 1\end{pmatrix} f(x) = \exp\{\pi i\, xBx^t\} f(x). \tag{3.7}$$

Remark. After an appropriate normalization of operators $\mathrm{We}(g)$, we can obtain

compact-sp
$$W(g)\mathcal{I}_{\mathbb{Z}_p^n} = \mathcal{I}_{\mathbb{Z}_p^n}, \qquad \text{where } g \in \mathrm{Sp}(2n,\mathbb{Z}_p)\ , \tag{3.8}$$

see also Proposition 4.3

3.5. Adelic Weil representation. We have

$$L^2(\mathbb{A}^n) = \bigotimes_p \Bigl(L^2(\mathbb{Q}_p^n), \mathcal{I}_{\mathbb{Z}_p^n}\Bigr), \qquad \mathcal{B}(\mathbb{A}^n) = \bigotimes_p \Bigl(L^2(\mathbb{Q}_p^n), \mathcal{I}_{\mathbb{Z}_p^n}\Bigr),$$

in the first case we have a tensor product in the category of Hilbert spaces, in the second case we have a tensor product in the category of abstract linear spaces.

Remark. To define a tensor products of an infinite family of spaces V_j, we need in a distingueshed unit vector e_j in each space, the tensor product space $\bigotimes V_j$ is spanned by products $v_1 \otimes v_2 \otimes \dots$, where $v_j = e_j$ for all j except a finite set. $\square$

The Weil representation of $\mathrm{Sp}(2n,\mathbb{A})$ is defined as $W(g) = \bigotimes W(g^{(p)})$. These operators are unitary in $L^2(\mathbb{A}^n)$ and preserve the dense subspace $\mathcal{B}(\mathbb{A}^n)$. For almost all $\mathcal{I}_{\mathbb{Z}_p^n}$, we have $\mathrm{We}(g^{(p)})\mathcal{I}_{\mathbb{Z}_p^n} = \mathcal{I}_{\mathbb{Z}_p^n}$ and this allows to define tensor products of operators.

3.6. Proof of Theorem 1.4.b. Transfer the representations of $\mathrm{Sp}(2n,\mathbb{Q})$ from the spaces $\mathcal{B}(\mathbb{A}^n)$, $\mathcal{M}(\mathbb{R}^n)$ to the space $\mathcal{M}(\mathbb{Q}^n)$. We obtain two representations of $\mathrm{Sp}(2n,\mathbb{Q})$ in $\mathcal{M}(\mathbb{Q}^n)$, say $\mathrm{We}_1(g)$, $\mathrm{We}_2(g)$. These operators satisfy the commutation relations

$$T(\sigma(h)) = \mathrm{We}_1(g)^{-1}T(h)\mathrm{We}_1(g), \qquad T(\sigma(h)) = \mathrm{We}_2(g)^{-1}T(h)\mathrm{We}_2(g).$$

Hence $\mathrm{We}_1(g)^{-1}\mathrm{We}_2(g)$ commutes with $T(h)$. By Lemma 2.2, $\mathrm{We}_2(g) = \lambda(g)\mathrm{We}_1(g)$, where $\lambda \in \mathbb{C}$. $\square$

4. Addendum. Constructions of modular forms

Here we explain the standard construction of modular forms from theta-functions and Howe duality, see [9], [1], [5], [4].

4.1. Congruence subgroups. Consider the group $\mathrm{Sp}(2n,\mathbb{Z})$ of symplectic matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with integer elements. For a positive integer N, denote by Γ_N the *principal congruence-subgroup* consisting of matrices $g \in \mathrm{Sp}(2n,\mathbb{Z})$ such that N divides all the matrix elements of $g-1$. A *congruence subgroup* in $\mathrm{Sp}(2n,\mathbb{Z})$ is any subgroup including a principal congruence-subgroup.

For the following statement, see, for instance, [10]

th:generators **Theorem 4.1.** *The subgroup in $U_l \subset \mathrm{Sp}(2n,\mathbb{Z})$ generated by matrices*

generators-spnz
$$\begin{pmatrix} 1+l\alpha & 0 \\ 0 & (1+l\alpha)^{t-1} \end{pmatrix}, \begin{pmatrix} 1 & l\beta \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ l\gamma & 1 \end{pmatrix}, \tag{4.1}$$

where α, β, γ, $(1+l\delta)^{-1}$ are integer matrices, is a congruence subgroup

4.2. The subgroup $\Gamma_{1,2}$. The denote by $\Gamma_{1,2}$ the subgroup of $\mathrm{Sp}(2n,\mathbb{Z})$ consisting of $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that the matrices A^tC and B^tD have even elements on the diagonals. For the following theorem, see [5]).

Theorem 4.2. *The group $\Gamma_{1,2}$ is generated by matrices*

$$\begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix},$$

where the matrices B, C have even diagonals.

Denote

$$\Delta(x) = \sum_{k_1,\dots,k_n} \prod_j \delta(x_j - \sqrt{2\pi} k_j)$$

pr:gamma-12 **Proposition 4.3.** *The restriction of the Weil representation of* $\mathrm{Sp}(2n,\mathbb{R})$ *to* $\Gamma_{1,2}$ *is a linear representation. Moreover, we can normalize the operators* $\mathrm{We}(g)$, $g \in \Gamma_{1,2}$, *such that*

we-delta
$$\mathrm{We}(g)\Delta = \Delta \tag{4.2}$$

PROOF. First, Δ is an eigenvector for operators $\mathrm{We}(g)$, $g \in \Gamma_{1,2}$. It is easy to verify this for generators of $\Gamma_{1,2}$, and hence this is valid for all g. Now we can choose the normalization (4.2). Now $\mathrm{We}(g)$ became a linear representation of $\Gamma_{1,2}$. □

4.3. Congruence subgroups and the space $\mathcal{P}(\mathbb{R}^n)$.

th:modular-origin **Theorem 4.4.** *The stabilizer of each element of $\mathcal{P}(\mathbb{R}^n)$ in the group $\Gamma_{1,2}$ is a congruence subgroup.*

PROOF. It is easy to verify (see Theorem 4.1) that the subgroup U_{2N^2} fix all the vectors of $\mathcal{P}(N^{-1}\mathbb{Z}^n | NZ^n)$. □

4.4. Modular forms of the weight 1/2. Denote by W_n the Siegel upper half-plane, i.e., the set of $n\times n$ complex matrices satisfying the condition $\frac{1}{2i}(z-z^*) > 0$. The group $\mathrm{Sp}(2n,\mathbb{R})$ acts in the space of holomorphic functions on W_n by the following operators

1/2
$$T_{1/2} \begin{pmatrix} A & B \\ C & D \end{pmatrix} f(x) = f((A+zC)^{-1}(B+zD)) \det(A+zC)^{-1/2} \tag{4.3}$$

Consider the operator

$$J\chi(z) = \{\exp(\frac{1}{2}xzx^t), \chi\}$$

from $\mathcal{S}'(\mathbb{R}^n)$ to our space of holomorphic functions. It is easy to verify, that this operator intertwines the Weil representation and the representation $T_{1/2}$.

By Proposition 4.3, for $g \in \Gamma_{1,2}$, we can normalize the operators (4.3)

$$T'_{1/2}(g) = \lambda(g) T_{1/2}(g), \qquad \lambda(g) \in \mathbb{C}$$

and obtain a linear representation of $\Gamma_{1,2}$ (in fact, $\lambda(g)$ ranges in 8-th roots of 1).

Proposition 4.5. *Let $\chi \in \mathcal{P}(\mathbb{R}^n)$ be a Poisson distribtion, $\Phi = J\chi$. There is a congruence subgroup $\Gamma \subset \Gamma_{12}$ such that*

$$T'_{1/2}(g)\Phi = \Phi, \qquad \text{where } g \in \Gamma$$

In fact, Theorem 4.4 provides lot of possibilities to produce modular forms. For instance, consider some embedding $I : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R})$ such that $i(\mathrm{SL}(2, \mathbb{Q})) \subset \mathrm{Sp}(2n, \mathbb{Q})$. Assume that the restriction of the Weil represntation to $\mathrm{SL}(2, \mathbb{R})$ contains a subrepresentation V of a discrete series[3]. Then we can consider projection of the space $\mathcal{P}(\mathbb{R}^n)$ to V.

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[3] On representations of $\mathrm{SL}(2, \mathbb{R})$, see, for instance [2].